Hermite and Laguerre β-ensembles: asymptotic corrections to the eigenvalue density

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Abstract

We consider Hermite and Laguerre β-ensembles of large $N \times N$ random matrices. For all β even, corrections to the limiting global density are obtained, and the limiting density at the soft edge is evaluated. We use the saddle point method on multidimensional integral representations of the density which are based on special realizations of the generalized (multivariate) classical orthogonal polynomials. The corrections to the bulk density are oscillatory terms that depends on β. At the edges, the density can be expressed as a multiple integral of the Konstevich type which constitutes a β-deformation of the Airy function. This allows us to obtain the main contribution to the soft edge density when the spectral parameter tends to $\pm\infty$.

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Contents

1 Introduction

We deal with two families of $N \times N$ random matrices: the Hermite and Laguerre β-ensembles (for a review see [9]). These ensembles possess an eigenvalue joint probability density function (p.d.f.) of the form

$$P_{N,\beta}(\mathbf{x}) = \frac{1}{Z_N} e^{-\beta W(\mathbf{x})}, \qquad \mathbf{x} = (x_1, \ldots, x_N) \in I^N, \tag{1}$$

where β is real and positive. The support I of the eigenvalues in the Hermite and Laguerre cases are respectively $(-\infty, \infty)$ and $(0, \infty)$. The ensembles' names come from the fact that their p.d.f. generalize the weight functions related to the Hermite and Laguerre polynomials; that is,

$$W(\mathbf{x}) = \begin{cases} \dfrac{1}{2} \sum_{i=1}^{N} x_i^2 - \sum_{1 \le i < j \le N} \ln |x_i - x_j|, & \text{Hermite}, \\[3mm] \dfrac{1}{2} \sum_{i=1}^{N} x_i - \dfrac{a}{2} \sum_{i=1}^{N} \ln |x_i| - \sum_{1 \le i < j \le N} \ln |x_i - x_j|, & \text{Laguerre}, \end{cases} \tag{2}$$

where a is a real and nonnegative parameter. The normalization constants can be computed with the help of the Selberg integrals:

$$Z_N = \begin{cases} G_{\beta,N} := g_{\beta,N} \displaystyle\prod_{j=2}^{N} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)}, & \text{Hermite}, \\[5mm] W_{a,\beta,N} := w_{a,\beta,N} \displaystyle\prod_{j=1}^{N} \frac{\Gamma(1 + j\beta/2)\Gamma(1 + (a + j - 1)\beta/2)}{\Gamma(1 + \beta/2)} & \text{Laguerre}, \end{cases} \tag{3}$$

where $g_{\beta,N} = (2\pi)^{N/2} \beta^{-N(1/2 + \beta(N-1)/4)}$ and $w_{a,\beta,N} = (2/\beta)^{N(a\beta/2 + 1 + \beta(N-1)/2)}$.

For special values of the Dyson index β, we recover classical random matrix ensembles (see e.g. [9, 19]). Indeed, the $\beta = 1, 2$, and 4 Hermite ensembles are respectively equivalent to the

Gaussian orthogonal, unitary, and symplectic ensembles. The Laguerre ensembles are similarly related to the real, complex and quaternionic Wishart matrices. Recently, Dumitriu and Edelman [3] have constructed explicit random matrices associated to the Hermite and Laguerre p.d.f. given in Eq.(1). A generic random $N \times N$ matrix belonging to the Hermite β-ensemble can be written as a tridiagonal symmetric matrix:

$$
H_\beta = \frac{1}{\sqrt{\beta}} \begin{pmatrix} \mathrm{N}[0,1] & \chi_{(N-1)\beta} & & & & \\ \chi_{(N-1)\beta} & \mathrm{N}[0,1] & \chi_{(N-2)\beta} & & & \\ & \chi_{(N-2)\beta} & \mathrm{N}[0,1] & \chi_{(N-3)\beta} & & \\ & \ddots & \ddots & \ddots & & \\ & & \chi_{2\beta} & \mathrm{N}[0,1] & \chi_\beta \\ & & & \chi_\beta & \mathrm{N}[0,1] \end{pmatrix}.
$$

This means that the N diagonal elements and the $N - 1$ subdiagonal elements are mutually independent; the diagonal elements are normally distributed (with mean zero and variance 1) while the off-diagonal have a chi distribution. Recall that the densities associated to $\mathrm{N}[\mu, \sigma]$ and χ_k are respectively $(2\pi\sigma^2)^{-1/2} e^{-(x-\mu)^2/(2\sigma^2)}$ and $2x^{k-1} e^{-x^2}/\Gamma(k/2)$, where in the latter case $x > 0$. Any $N \times N$ matrix L_β of the Laguerre β-ensemble also has a tridiagonal form: $L_\beta = B_\beta^T B_\beta$, for some $N \times N$ matrix

$$
B_\beta = \frac{1}{\sqrt{\beta}} \begin{pmatrix} \chi_{P\beta} & \chi_{(N-1)\beta} & & \\ & \chi_{(P-1)\beta} & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \\ & & \chi_{(P-N+1)\beta} & \chi_\beta \end{pmatrix}, \qquad a = P - N + 1 - \frac{2}{\beta}.
$$

In this article, we compute the density for large but finite random matrices of the Hermite and Laguerre β-ensembles. The density, or the marginal eigenvalue probability density, is defined as follows:

$$
\rho_{N,\beta}(x) := \frac{N}{Z_N} \int_{I^N} P_{N,\beta}(x_1, \ldots, x_N)\, dx_1 \cdots dx_N. \tag{4}
$$

The quantity $N^{-1}\rho_{N,\beta}(x)dx$ represents the probability to have an eigenvalue in the interval $[x, x + dx]$. The density has two simple physical interpretations.

First, we remark that the Hermite p.d.f. is equivalent to the Boltzmann factor of a log-potential Coulomb gas with particles of charge unity confined to the interval $(-\sqrt{2N}, \sqrt{2N})$ with neutralizing background charge density $-(\sqrt{2N}/\pi)\sqrt{1 - x^2/2N}$. From this point of view, Z_N (divided by $N!$) is simply the canonical partition function at inverse temperature β and $\rho_{N,\beta}(x)dx$ gives the number of charges present in the interval $[x, x + dx]$. This analogy allows one to predict the global density :

$$
\lim_{N\to\infty} \sqrt{\frac{2}{N}} \rho_{N,\beta}(\sqrt{2N}x) = \rho_{\mathrm{W}}(x) := \begin{cases} \dfrac{2}{\pi}\sqrt{1 - x^2}, & -1 < x < 1, \\ 0, & |x| \geq 1. \end{cases} \tag{5}
$$

This result is known as the Wigner semicircle law. For a finite matrix, we expect that the scaled density is of order one in the interval $(-\sqrt{2N}, \sqrt{2N})$, the 'bulk region' of the mechanical problem, while it decreases rapidly around $\pm\sqrt{2N}$, called the 'soft edges'. A similar log-gas

construction is possible for the Laguerre case. One expects the $c = 1$ Marčenko-Pastur law [18]:

$$\lim_{N \to \infty} 4\rho_{N,\beta}(4Nx) = \rho_{\mathrm{MP}}(x) := \begin{cases} \dfrac{2}{\pi}\sqrt{\dfrac{1}{x} - 1}, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases} \tag{6}$$

We see that, in the Laguerre case, the 'bulk' is $(0, 4N)$ while the 'soft edge' is the point $4N$. The origin is referred as the 'hard edge' of the support because the eigenvalues are constrained to be positive. The predictions given in Eqs (5) and (6) have been confirmed in [1, 7]. The asymptotic analysis used in these references constitutes the starting point for the study of the higher expansions to be undertaken in the present work.

Second, their is a deep connection between the β-ensembles and some integrable quantum mechanical N-body problems on the line, known as the Calogero-Moser-Sutherland (CMS) models (a good reference is [21]). The Hermite p.d.f. is in fact the ground state wave functions squared of the (rational) A_{N-1} CMS model, whose Hamiltonian is

$$H^{(\mathrm{H})} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\beta^2}{4}\sum_{i=1}^{N} x_i^2 + \frac{\beta(\beta-2)}{2} \sum_{1 \leq i < j \leq N} \frac{1}{(x_i - x_j)^2},$$

for $x_j \in (-\infty, \infty)$. The Laguerre p.d.f. is the ground state squared of the Hamiltonian of the B_N CMS model, which can be expressed as follows:

$$\begin{aligned} H^{(\mathrm{L})} = &- 2\sum_{i=1}^{N}\left(2x_i \frac{\partial^2}{\partial x_i^2} + \frac{\partial}{\partial x_i}\right) \\ &+ \frac{1}{4}\sum_{i=1}^{N}\left(a\beta(a\beta - 2)\frac{1}{x_i} + \beta^2 x_i\right) + \beta(\beta - 2)\sum_{1 \leq i < j \leq N}\frac{x_i + x_j}{(x_i - x_j)^2}, \end{aligned}$$

where $x_j \in (0, \infty)$. It has been shown in [1] (see also [23]) that the eigenfunctions of the conjugated Schrödinger operators $e^{\beta W/2}H^{(\mathrm{H})}e^{-\beta W/2}$ and $e^{\beta W/2}H^{(\mathrm{L})}e^{-\beta W/2}$ are respectively the generalized (or multivariate) Hermite and Laguerre polynomials, previously introduced by Lassalle in [17, 16]. In the context of CMS models, the global density can be seen as the ground state expectation value of the density operator $\hat{\rho}(x) = \sum_{j=1,\ldots,N}\delta(x - x_j)$, also known as the one-point function.

The relation between the CMS models and the generalized classical orthogonal polynomials furnishes, when β is an even integer, new integral representations of the global density that suits perfectly for asymptotic analysis. Let us be more explicit. The definition of the density given in (4) contains N integrals; considering N large does not simplify the calculation. On the other hand, it has been noticed in [1, 7] that the density is a particular Hermite (or Laguerre) polynomial, characterized by a partition $\lambda = ((N-1)^\beta)$ and evaluated at $x_1 = \ldots = x_\beta = x$ (see below). Using the work of Kaneko [14] and Yan [22], one then can shows that the density is proportional to the following β-dimensional integral:

$$R_{N,\beta}(x) := \int_{\mathcal{C}} du_1\, e^{Nf(u_1,x)} \cdots \int_{\mathcal{C}} du_\beta\, e^{Nf(u_\beta,x)} \prod_{1 \leq j < k \leq \beta} |u_j - u_k|^{4/\beta}, \tag{7}$$

for a particular contour $\mathcal{C}$ and function $f(u, x)$.

In the following sections, we apply the steepest descent method [20, 24] to integrals of the type (7). We obtain expressions for the density in the bulk and at the soft edge that are valid for every $\beta \in 2\mathbb{N}$. Of course, these results generalize many known result obtained for $\beta = 2$ and 4. We mention in particular two recent publications in which asymptotic corrections to the global density have been obtained: 1) Kalisch and Braak [13] for some ensembles, including the Gaussian unitary and symplectic ensembles (work based on the supersymmetric method); 2) Garoni, Frankel and Forrester [11] for the Laguerre and Gaussian unitary ensembles (calculations using the theory of orthogonal polynomials). Also, the preprint [10] of Forrester, Frankel and Garoni addresses the Laguerre and Gaussian ensembles with orthogonal and symplectic symmetry. All studies show that these approximate expressions of the global density are very accurate, even for $N = 10$, say (for instance, see Fig. 1 and Fig. 2 in [11]). We finally point out that an asymptotic formula for the density in the Hermite β-ensemble has been considered in a different context: Johansson [12] has studied a smoothed (macroscopic) density and has derived corrections of order $1/N$ to Eq. (5). However, contrary to the asymptotic formula obtained here, the large N expansion given in [12] does not contain oscillatory (microscopic) terms.

The article is organized as follows. In Section 2, we review the exact expressions of the densities in terms of the generalized Hermite and Laguerre polynomials. In Section 3, we derive the first oscillatory corrections to the global densities (5) and (6). These approximations are also compared to the exact densities given in Section 2. The asymptotic densities evaluated about the soft edges of the spectrum are obtained in Section 4; they are expressed in terms of Kontsevich type integrals. The behavior of the latter when the spectral parameter is large is studied in Section 4. In the last section, we finally summarize the principal results and discuss the generalization of some of our results to general β.

2   Exact expressions of the density

As previously mentioned, the density in the Hermite and Laguerre ensembles can be written as particular generalized Hermite and Laguerre polynomials [1]. These polynomials are symmetric, so we can write them as a linear combination of monomial symmetric functions

$$m_\lambda(x_1, \ldots, x_N) := x_1^{\lambda_1} \cdots x_N^{\lambda_N} + \text{distinct permutations},$$

where $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a partition of weight $|\lambda| = \sum_{i=1}^{N} \lambda_i$. It is convenient to introduce another basis of the algebra of symmetric polynomials, namely, the (monic) Jack polynomials $\bar{J}_\lambda^{(\alpha)}$. They constitute the only basis such that

$$\bar{J}_\lambda^{(\alpha)}(x_1, \ldots, x_N) = m_\lambda(x_1, \ldots, x_N) + \sum_{\mu < \lambda} a_{\lambda\mu}(\alpha) \, m_\mu(x_1, \ldots, x_N) \qquad \text{(triangularity)}$$

$$D_2^{(\alpha)} \, \bar{J}_\lambda^{(\alpha)}(x_1, \ldots, x_N) = \epsilon_\lambda(\alpha) \, \bar{J}_\lambda^{(\alpha)}(x_1, \ldots, x_N) \qquad \text{(eigenfunction)}$$

for some eigenvalue $\epsilon_\lambda(\alpha)$. In the last equations, $\mu < \lambda$ means that $\sum_{j=1}^{k} \mu_i \leq \sum_{j=1}^{k} \lambda_i$ for all k when $|\mu| = |\lambda|$ but $\mu \neq \lambda$, while $D_2^{(\alpha)}$ is a particular differential operator that can be defined via

$$D_k^{(\alpha)} := \sum_{i=1}^{N} x_i^k \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{1 \leq i < j \leq N} \frac{1}{x_i - x_j} \left(x_i^k \frac{\partial}{\partial x_i} - x_j^k \frac{\partial}{\partial x_j} \right).$$

The generalized Hermite polynomials, denoted by $\bar{H}_\lambda(x_1, \ldots, x_N; \alpha)$, are the only symmetric polynomials obeying to

$$\bar{H}_\lambda(x_1, \ldots, x_N; \alpha) = \bar{J}_\lambda^{(\alpha)}(x_1, \ldots, x_N) + \sum_{\substack{|\mu|=|\lambda|-2n \\ n=1,2,\ldots,\lfloor|\lambda/2|\rfloor}} b_{\lambda\mu}(\alpha, N)\, \bar{J}_\mu^{(\alpha)}(x_1, \ldots, x_N),$$

$$(D_0^{(\alpha)} - 2E_1)\, \bar{H}_\lambda(x_1, \ldots, x_N; \alpha) = -2|\lambda|\, \bar{H}_\lambda(x_1, \ldots, x_N; \alpha),$$

where

$$E_k := \sum_{i=1}^N x_i^k \frac{\partial}{\partial x_i}.$$

Let us point out that

$$D_0^{(\alpha)} - 2E_1 = -\frac{2}{\beta} e^{\beta W/2} H^{(\mathrm{H})} e^{-\beta W/2} + \mathrm{cst}, \qquad \beta = \frac{2}{\alpha},$$

where $H^{(\mathrm{H})}$ is the CMS Hamiltonian defined in Section 1. On can show that

$$\bar{H}_\lambda(x_1, \ldots, x_N; \alpha) = \exp\left(-\frac{1}{4} D_0^{(\alpha)}\right) \bar{J}_\lambda^{(\alpha)} \tag{8}$$

Similarly to the Hermite case, the generalized Laguerre polynomials, written $\bar{L}_\lambda^\nu(x_1, \ldots, x_N; \alpha)$, are the unique symmetric polynomials satisfying

$$\bar{L}_\lambda^\nu(x_1, \ldots, x_N; \alpha) = \bar{J}_\lambda^{(\alpha)}(x_1, \ldots, x_N) + \sum_{\substack{|\mu|=|\lambda|-n \\ n=1,2,\ldots,|\lambda|}} c_{\lambda\mu}(\alpha, \nu, N)\, \bar{J}_\mu^{(\alpha)}(x_1, \ldots, x_N),$$

$$(D_1^{(\alpha)} - E_1 + (\nu+1)E_0)\, \bar{L}_\lambda^\nu(x_1, \ldots, x_N; \alpha) = -|\lambda|\, \bar{L}_\lambda^\nu(x_1, \ldots, x_N; \alpha).$$

The latter eigenvalue problem is related to a CMS model:

$$D_1^{(\alpha)} - E_1 + (\nu+1)E_0 = -\frac{1}{2\beta} e^{\beta W/2} H^{(\mathrm{L})} e^{-\beta W/2} + \mathrm{cst}, \qquad \beta = \frac{2}{\alpha}, \qquad \nu = \frac{\beta a - 1}{2}.$$

The following formula furnishes a way to compute the generalized Laguerre polynomials:

$$\bar{L}_\lambda^\nu(x_1, \ldots, x_N; \alpha) = \exp\left(-D_1^{(\alpha)} - (\nu+1)E_0\right) \bar{J}_\lambda^{(\alpha)}(x_1, \ldots, x_N). \tag{9}$$

When β is an even integer, the density in the Hermite and Laguerre ensembles can be respectively written as a particular Hermite and Laguerre polynomial; explicitly,

$$\rho_{N,\beta}(x) = \begin{cases} N \dfrac{G_{\beta,N-1}}{G_{\beta,N}} e^{-\beta x^2/2} \bar{H}_{((N-1)^\beta)}(x_1, \ldots, x_\beta; \beta/2)\big|_{x_1=\ldots=x_\beta=x}, & \text{Hermite,} \\[2ex] N \dfrac{W_{a,\beta,N-1}}{W_{a,\beta,N}} x^{a\beta/2} e^{-\beta x/2} \bar{L}_{((N-1)^\beta)}^{a-1+2/\beta}(x_1, \ldots, x_\beta; \beta/2)\big|_{x_1=\ldots=x_\beta=x}, & \text{Laguerre,} \end{cases} \tag{10}$$

where we have used the convention $(n^k) = \overbrace{(n, \ldots, n)}^{k}$. Eqs (8) and (9), together with the fact that $\bar{J}_\lambda^{(\alpha)}(x_1, \ldots, x_k) = x_1^n \cdots x_k^n$ when $\lambda = (n^k)$, readily imply the following exact expressions of the density:

$$\rho_{N,\beta}(x) = N \frac{G_{\beta,N-1}}{G_{\beta,N}} e^{-\beta x^2/2} \sum_{n=0}^{\beta(N-1)/2} \left[\frac{(-1)^n}{4^n n!} \left(D_0^{(\beta/2)}\right)^n x_1^{N-1} \cdots x_\beta^{N-1}\right]_{x_1=\ldots=x_\beta=x} \tag{11}$$

in the Hermite case, and

$$\rho_{N,\beta}(x) = N\frac{W_{a,\beta,N-1}}{W_{a,\beta,N}}x^{a\beta/2}e^{-\beta x/2}$$

$$\times \sum_{n=0}^{\beta(N-1)}\left[\frac{(-1)^n}{n!}\left(D_1^{(\beta/2)}+(a-2/\beta)E_0\right)^n x_1^{N-1}\cdots x_\beta^{N-1}\right]_{x_1=\ldots=x_\beta=x}, \quad (12)$$

in the Laguerre case.

We will use the two latter formulas to compare the exact and asymptotic expressions of the density. Note however that Eqs (11) and (12) are computable only when both N and β are small (i.e., for small partitions in Eq. (10)). There exist other methods that allow to calculate the multivariate classical polynomials (see for instance [5]), but they suffer from the same restrictions. As shown in the next section, the asymptotic expressions provide a more tractable way to determine the density when N is large.

3   Density in the bulk

In this section, we obtain oscillatory corrections to the global densities (5) and (6). This is achieved by deforming the contours of integration $\mathcal{C}$ in (7) in such a way that they pass through the saddle points of the function $f(u,x)$. In both the Hermite and the Laguerre cases, the function $f(u,x)$ has two simple saddle points in the complex u-plane, called u_+ and u_-. All oscillatory terms can be seen as combinatorial corrections: the global density is recovered when $\beta/2$ variables go through u_+ while the remaining $\beta/2$ variables go through u_-; the dominant oscillatory term comes from the integration of $\beta/2+1$ variables through u_+ and $\beta/2-1$ variables through u_- and conversely; the second oscillatory term comes the integration of $\beta/2+2$ variables through u_+ and $\beta/2-2$ variables through u_- and conversely; and so on.

3.1   Hermite case

Before considering explicitly the density in the Hermite ensemble, we prove two technical results associated to the asymptotics of the integral (7). In the following lines, we suppose that $e^{Nf(u,x)}$ is analytic everywhere in the finite complex u-plane, except possibly at a pole depending on x, and that $\mathcal{C}$ is the real interval $(-\infty,\infty)$.

Figure 1: New contours $\{\mathcal{C}_1,\ldots,\mathcal{C}_n\}$ in the complex u_j-plane.
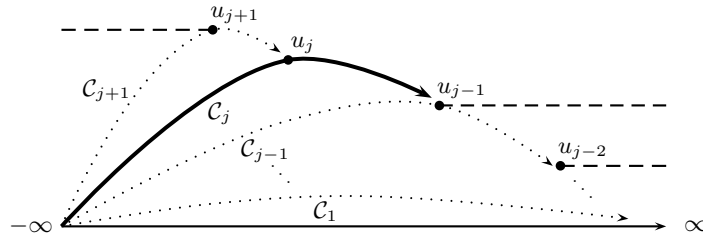
The method of steepest descent requires that the integrand of (7) should be analytic. This means in particular that the absolute values must be removed. Such an operation is realized in

the following lemma; it is possible when the line integration $\mathcal{C}$ of the variable u_j is deformed into an appropriate complex path $\mathcal{C}_j$. Acceptable contours are given in Fig. 1. Other appropriate contours are obtained by making a reflection of the picture with respect to the real axis. Note that the dashed lines stand for (movable) branch cuts. We stress that u_j's contour starts at $-\infty$ and ends at the complex variable u_{j-1}. Only the path of u_1 (the last variable to be integrated) ends on the real axis.

Lemma 1. *Let $\{\mathcal{C}_j\}$ be a set of non-intersecting contours such that $\mathcal{C}_1$ is a simple contour going from $-\infty$ to ∞ and such that $\mathcal{C}_j$ goes from $-\infty$ to u_{j-1} for all $j = 2, \ldots, n$ (see Fig. 1). Then*

$$\int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n \prod_{i=1}^{n} e^{Nf(u_i,x)} \prod_{1 \le j < k \le n} |u_j - u_k|^{4/\beta} =$$

$$n! \int_{\mathcal{C}_1} du_1 \cdots \int_{\mathcal{C}_n} du_n \prod_{i=1}^{n} e^{Nf(u_i,x)} \prod_{1 \le j < k \le n} (u_j - u_k)^{4/\beta},$$

where $-\pi < \arg u_j \le \pi$ and where $\arg(u_i - u_j)^{4/\beta} = 0$ when $u_i, u_j \in \mathbb{R}$ but $u_i > u_j$.[1]

Proof. Using the invariance of the lefthand side of the previous equation under any permutation, we immediately see that it is equivalent to the following ordered integrals:

$$n! \int_{-\infty}^{\infty} du_1 \, e^{Nf(u_1,x)} \int_{-\infty}^{u_1} du_2 \, e^{Nf(u_2,x)} \cdots \int_{-\infty}^{u_{n-1}} du_n \, e^{Nf(u_n,x)} \prod_{1 \le j < k \le n} (u_j - u_k)^{4/\beta} \,.$$

These integrals only contain analytic functions, so we can use Cauchy's Theorem. This implies that the contour of u_1 can be deformed into any simple curve starting at $-\infty$ and stopping at ∞. For the remaining variables, any nonintersecting contour of integration in the complex plane which starts at $-\infty$, doesn't cross the branch cuts coming coming from the multivaluedness of the integrand, and complies with the ordering of the variables can be chosen. $\qquad \square$

We are now in position to analyse (7) when N is large. Recall that the basic idea of the steepest descent method is to choose a path for which the decrease of $f(u,x)$ in maximum. In particular, this means that the contour must pass through the saddle points. In the bulk case, f has two simple saddle points $u_\pm$; that is,

$$\frac{\partial}{\partial u} f(u,x)\big|_{u_\pm} = 0, \qquad \frac{\partial^2}{\partial u^2} f(u,x)\big|_{u_\pm} = Re^{i\phi_\pm}, \quad R > 0. \tag{13}$$

The directions of steepest descent at these points, denoted $\theta_\pm$, are such that $\cos(2\theta_\pm + \phi_\pm) = -1$ and $\sin(2\theta_\pm + \phi_\pm) = 0$, so

$$\theta_\pm = \frac{\pi - \phi_\pm}{2} \pmod{\pi}, \qquad -\pi < \theta_\pm \le \pi. \tag{14}$$

---

[1]When $n \ne \beta$ and $\beta \notin \mathbb{N}$ in Lemma 1, other integral representations in which all variables go from $-\infty$ to ∞ are possible [2].

Proposition 2. *Let $f(u,x)$ be a function that satisfies Eqs (13) and (14). Let also $f_\pm = f(u_\pm, x)$. Suppose moreover that the saddle points are such that $\Re(u_-) < \Re(u_+)$. Then,*

$$
R_{N,\beta}(x) = \binom{\beta}{\beta/2}(\Gamma_{\beta/2,\beta})^2(u_+ - u_-)^\beta \left(\frac{2}{NR}\right)^{\beta-1}
$$

$$
\times\, e^{\beta N(f_+ + f_-)/2} e^{\mathrm{i}(\beta-1)(\theta_+ + \theta_-)} \left[r_{N,\beta}(x) + \mathrm{O}\left(\frac{1}{N}\right) \right],
$$

where

$$
r_{N,\beta}(x) = 1 + 2 \sum_{k=1}^{\lfloor \sqrt{\beta/2} \rfloor} \left[\left(\prod_{j=1}^{k} \frac{\Gamma(1+2j/\beta)}{\Gamma(1+2(j-k)/\beta)} \right) \right.
$$

$$
\left. \times\, \frac{e^{\mathrm{i}2k^2(\theta_+ + \theta_-)/\beta}}{(u_+ - u_-)^{4k^2/\beta}(NR)^{2k^2/\beta}} \cos\left(-\mathrm{i}kN(f_+ - f_-) + k(\theta_+ - \theta_-)(3 - 2/\beta) \right) \right]
$$

and

$$
\Gamma_{n,\beta} := \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n \prod_{i=1}^{n} e^{-u_i^2} \prod_{1 \le j < k \le n} |u_j - u_k|^{4/\beta} = \frac{\pi^{n/2}}{2^{n(n-1)/\beta}} \prod_{j=2}^{n} \frac{\Gamma(1+2j/\beta)}{\Gamma(1+2/\beta)}.
$$

$$(15)$$

Proof. We first apply Lemma 1 to the expression (7). Then, the contours $\mathcal{C}_j$ are deformed into steepest descent contours $\mathcal{S}_j = \mathcal{S}_j^- \cup \mathcal{S}_j^+$ passing through the saddle points $u_\pm$. Close to these points, the contours are parametrized as follows:

$$
u_j = u_\pm + t_j e^{\mathrm{i}\theta_\pm}, \quad -\pi < \arg u_j \le \pi, \quad \text{on} \quad \mathcal{S}_j^\pm,
$$

where the angles of steepest descent are given (14) and where $t_j \in (-\tau, \tau)$ for some $\tau > 0$. Moreover, we impose $t_i > t_j$ for $i < j$ in order to guarantee $\Re(u_i) > \Re(u_j)$ for $i < j$. Setting

$$
y_j = \sqrt{\frac{NR}{2}} t_j
$$

we obtain

$$
y_i \in (-\infty, \infty) \quad \text{and} \quad f(u_j, x) = N f_\pm - y_j^2 + \mathrm{O}(1/\sqrt{N})
$$

as $N \to \infty$. When both u_j and u_k are close to the same saddle point $u_\pm$,

$$
(u_j - u_k)^{4/\beta} = \left(\frac{2}{NR}\right)^{2/\beta} e^{\mathrm{i}4\theta_\pm/\beta}(y_j - y_k)^{4/\beta}.
$$

When u_j is on $\mathcal{S}_j^+$ while u_k is on $\mathcal{S}_j^-$, we have

$$
(u_j - u_k)^{4/\beta} = (u_+ - u_-)^{4/\beta} + \mathrm{O}(1/\sqrt{N}).
$$

We now return to the expression of $R_{N,\beta}(x)$ by considering the steepest descent paths:

$$R_{N,\beta}(x) = \beta! \sum_{n=0}^{\beta} \left(\int_{\mathcal{S}_j^+} du_1 \cdots \int_{\mathcal{S}_j^+} du_n \right.$$
$$\left. \times \int_{\mathcal{S}_j^-} du_{n+1} \cdots \int_{\mathcal{S}_j^-} du_\beta \prod_{j=1}^{n} e^{Nf(u_j,x)} \prod_{1 \le j < k \le \beta} (u_j - u_k)^{4/\beta} \right).$$

In terms of the new variables introduced above, the righthand side of the previous equation becomes

$$\beta! \sum_{n=0}^{\beta} S_n \left[\int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{y_1} dy_2 \cdots \int_{-\infty}^{y_{n-1}} dy_n \prod_{j=1}^{n} e^{-y_j^2} \left(1 + O\left(\frac{1}{\sqrt{N}} \right) \right) \prod_{1 \le p < q \le n} (y_p - y_q)^{4/\beta} \right.$$
$$\left. \times \int_{-\infty}^{\infty} dy_{n+1} \int_{-\infty}^{y_{n+1}} dy_{n+2} \cdots \int_{-\infty}^{y_{\beta-1}} dy_\beta \prod_{k=n+1}^{\beta} e^{-y_k^2} \left(1 + O\left(\frac{1}{\sqrt{N}} \right) \right) \prod_{n+1 \le r < s \le \beta} (y_r - y_s)^{4/\beta} \right],$$

where

$$S_n = (u_+ - u_-)^{4n(\beta-n)/\beta} \left(\frac{2}{NR} \right)^{3\beta/2 - 1 - 2n(\beta-n)/\beta}$$
$$\times e^{Nnf_+ + N(\beta-n)f_-} e^{i\theta_+ (n + 2n(n-1)/\beta)} e^{i\theta_- (\beta - n + 2(\beta-n)(\beta-n-1)/\beta)}.$$

The Gaussian terms are in fact ordered versions of the functions $\Gamma_{n,\beta}$, introduced in Eq.(15). We again use Lemma 1 and get

$$R_{N,\beta}(x) = \sum_{n=0}^{\beta} \binom{\beta}{n} \left[S_n \Gamma_{n,\beta} \Gamma_{\beta-n,\beta} + O\left(\frac{1}{N} \right) \right]$$
$$= \binom{\beta}{\beta/2} \left(\Gamma_{\beta/2,\beta} \right)^2 S_{\beta/2} \left[r_{N,\beta}(x) + O\left(\frac{1}{N} \right) \right],$$

where

$$r_{N,\beta}(x) = 1 + \sum_{k=1}^{\beta/2} \binom{\beta}{\beta/2 + k} \binom{\beta}{\beta/2}^{-1} \frac{\Gamma_{\beta/2+k,\beta} \Gamma_{\beta/2-k,\beta}}{\Gamma_{\beta/2,\beta} \Gamma_{\beta/2,\beta}} \left[\frac{S_{\beta/2+k} + S_{\beta/2-k}}{S_{\beta/2}} + O\left(\frac{1}{N} \right) \right].$$
$$(16)$$

Note that the order of the 'analytic corrections' (i.e., coming from the expansion of $f(u_j, x)$ as a polynomial in y_j of degree superior than 2) is now $1/N$ rather than $1/\sqrt{N}$. This can be explained by using the theory of the generalized Hermite polynomials [1]: only symmetric polynomials $p(y_1, \ldots, y_n)$ of degree even may have a nonzero contribution to

$$\int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \prod_{i=1}^{n} e^{-y_i^2} \prod_{1 \le j < k \le n} |y_j - y_k|^{4/\beta} p(y_1, \ldots, y_n).$$

In our case, the $O(1/\sqrt{N})$ terms are symmetric polynomials in y_j of degree one and three, so they don't contribute in the expression of $R_{N,\beta}(x)$.

Note also that $(S_{\beta/2+k} + S_{\beta/2-k})/S_{\beta/2}$ is of order $1/N^{2k^2/\beta}$. Hence, the 'combinatorial terms' with $k > \lfloor\sqrt{\beta/2}\rfloor$ are smaller than the 'analytic corrections' of order $1/N$; thus we must truncate Eq. (16) as follows:

$$r_{N,\beta}(x) = 1 + \sum_{k=1}^{\lfloor\sqrt{\beta/2}\rfloor} \binom{\beta}{\beta/2+k}\binom{\beta}{\beta/2}^{-1} \frac{\Gamma_{\beta/2+k,\beta}\Gamma_{\beta/2-k,\beta}}{\Gamma_{\beta/2,\beta}\Gamma_{\beta/2,\beta}} \left[\frac{S_{\beta/2+k} + S_{\beta/2-k}}{S_{\beta/2}}\right].$$

This expression contains only 'combinatorial corrections'.

Finally, one readily shows that

$$\binom{\beta}{\beta/2+k}\binom{\beta}{\beta/2}^{-1} \frac{\Gamma_{\beta/2+k,\beta}\Gamma_{\beta/2-k,\beta}}{\Gamma_{\beta/2,\beta}\Gamma_{\beta/2,\beta}} = \frac{1}{2^{2k^2/\beta}} \prod_{j=1}^{k} \frac{\Gamma(1+2j/\beta)}{\Gamma(1+2(j-k)/\beta)}$$

and the proof is complete.   □

The explicit link between the Hermite density and the integral of the type (7) has been obtained in [1]. It reads:

$$\rho_{N,\beta}(\sqrt{2N}x) = \frac{1}{2}\frac{G_{\beta,N-1}}{G_{\beta,N}\Gamma_{\beta,\beta}}(2N)^{\beta N/2+\beta}e^{-\beta Nx^2}R_{N,\beta}(x) \tag{17}$$

if

$$f(u,x) = -2u^2 + \ln(iu+x) - \frac{1}{N}\ln(iu+x). \tag{18}$$

Up to additive terms of order $1/N$, the latter function has two saddle points

$$u_\pm = \frac{1}{2}(ix \pm \sqrt{1-x^2}).$$

We see that $\Re(u_+) > \Re(u_-)$ only if $-1 < x < 1$. According to the notation used in Eqs (13) and (14), we have

$$f_\pm = -\frac{1}{2} - \frac{N-1}{N}\ln 2 + x^2 \pm i\left(\frac{N-1}{N}\arccos x - x\sqrt{1-x^2}\right)$$

and

$$Re^{i\phi_\pm} = 8\sqrt{1-x^2}e^{i(\pi-\arcsin x)},$$

where we have made use of

$$\arcsin x = -i\ln(ix + \sqrt{1-x^2}) = \pi/2 - \arccos x.$$

Note that the inverse trigonometric functions are defined on their principal branch; that is, $\arcsin x : [-1,1] \longrightarrow [-\pi/2, \pi/2]$ and $\arccos x : [-1,1] \longrightarrow [\pi, 0]$. Thus the angles of steepest descent are

$$\theta_\pm = \mp\frac{1}{2}\arcsin x.$$

Stirling's approximation,

$$\Gamma(y + z) = \sqrt{2\pi} e^{-z} z^{y+z-1/2} \left[1 + \mathrm{O}\left(\frac{1}{z}\right)\right] \quad \text{when} \quad z \to \infty,$$

immediately implies

$$\frac{G_{\beta,N-1}}{G_{\beta,N}} = \frac{2^{\beta N/2-1/2} \beta^{\beta N/2-\beta} e^{\beta N/2}}{\pi N^{\beta N/2+1/2}} \left[1 + \mathrm{O}\left(\frac{1}{N}\right)\right].$$

The substitution of the above results in Proposition 2 provide the sought asymptotic corrections to the global density.

Corollary 3. *Let $-1 < x < 1$ and let $P_W(x)$ denote the (cumulative) probability distribution associated to the semicircle law given in Eq. (5); i.e.,*

$$P_W(x) = \int_{-1}^{x} \rho_W(t) dt = 1 + \frac{x}{2}\rho_W(x) - \frac{1}{\pi}\arccos x. \tag{19}$$

Then

$$\sqrt{\frac{2}{N}}\rho_{N,\beta}(\sqrt{2N}x) = \rho_W(x)r_{N,\beta}(x) + \mathrm{O}\left(\frac{1}{N}\right),$$

where

$$r_{N,\beta}(x) = 1 + 2\sum_{k=1}^{\lfloor\sqrt{\beta/2}\rfloor} \frac{(-1)^k}{(\pi^3\rho_W(x)^3 N)^{2k^2/\beta}}$$

$$\times \left(\prod_{j=1}^{k} \frac{\Gamma(1 + 2j/\beta)}{\Gamma(1 + 2(j-k)/\beta)}\right) \cos\left(2\pi k N P_W(x) + k\varphi(x,\beta)\right) \tag{20}$$

for $\varphi(x,\beta) = \left(1 - \frac{2}{\beta}\right)\arcsin x.$

Let us consider only the very first correction to the global density:

$$\sqrt{\frac{2}{N}}\rho_{N,\beta}(\sqrt{2N}x) = \rho_W(x) + \mathrm{O}\left(\frac{1}{N}\right) + \mathrm{O}\left(\frac{1}{N^{8/\beta}}\right)$$

$$- \frac{2}{\pi} \frac{\Gamma(1 + 2/\beta)}{(\pi\rho_W(x))^{6/\beta-1}} \frac{1}{N^{2/\beta}} \cos\left(2\pi N P_W(x) + \varphi(x,\beta)\right). \tag{21}$$

Up to a factor of order $1/N$, the dominant oscillatory terms in the Gaussian unitary and symplectic ensembles are thus

$$\sqrt{\frac{2}{N}}\rho_{N,\beta}(\sqrt{2N}x) - \rho_W(x) = \begin{cases} \dfrac{-2}{\pi^3\rho_W(x)^2 N} \cos\left(2\pi N P_W(x)\right), & \beta = 2. \\[2ex] \dfrac{-1}{\pi\rho_W(x)^{1/2} N^{1/2}} \cos\left(2\pi N P_W(x) + \dfrac{1}{2}\arcsin x\right), & \beta = 4, \end{cases}$$

respectively. A direct computation shows that the non-oscillatory $O(1/N)$ term is exactly zero when $\beta = 2$. This implies that our result reproduces the Gaussian global densities previously obtained in [11, 13], for the unitary case, and in [13], for the symplectic case. [2]

The asymptotic expansion of the density for $\beta = 6$ is numerically compared with the exact one in Fig. 2. This picture shows that, even for a small N's, Eq. (20) furnishes a qualitatively good approximation of the density in the bulk. Fig. 3 illustrates the behavior of $\rho_{N,\beta}(\sqrt{2N}x)$ when the Dyson index varies.

Figure 2: Comparison of the exact density (11), shown as a solid line, and the asymptotic density (21), shown as a dashed line, in the Hermite β-ensemble for $N = 7$ and $\beta = 6$.
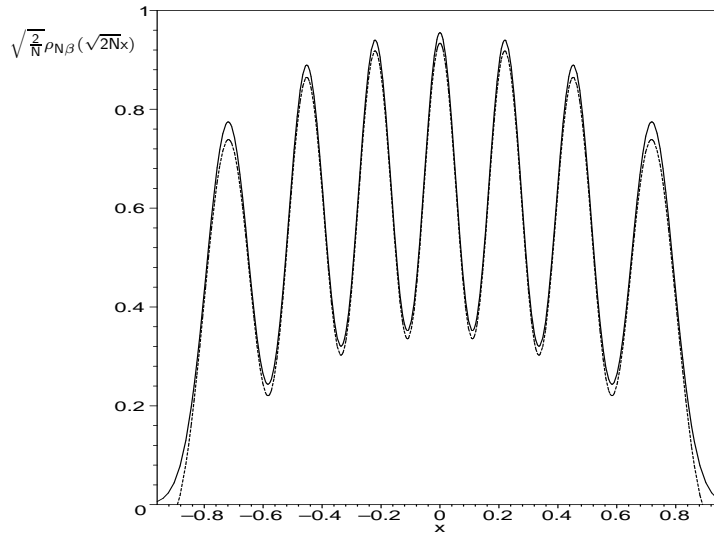

3.2 Laguerre case

The method used to evaluate the asymptotic behavior the the Laguerre density is almost the same as the one used in the Hermite case. All relevant differences originate from the correct contour that we must choose in Eq. 7 [7]: $\mathcal{C}$ starts at the point $u = 1$, turns around zero in the counterclockwise direction, and comes back to $u = 1$. In the following paragraphes, we briefly obtain the Laguerre version of Lemma 1 and Proposition 2. We suppose that $e^{Nf(u,x)}$ is analytic everywhere, except maybe on the interval $[0, 1]$.

Lemma 4. *Let $\{\mathcal{C}_j\}$ be a set of nonintersecting counterclockwise contours around the origin, all starting at $u_j = 1$, such that $0 \leq \arg(u_n) \leq \ldots \leq \arg(u_1) \leq 2\pi$. Then*

$$\oint_{\mathcal{C}} du_1 \cdots \oint_{\mathcal{C}} du_n \prod_{i=1}^{n} e^{Nf(u_i,x)} \prod_{1 \leq j < k \leq n} |u_j - u_k|^{4/\beta} =$$

$$n!(-1)^{n(n-1)/\beta} \int_{\mathcal{C}_1} du_1 \cdots \int_{\mathcal{C}_n} du_n \prod_{i=1}^{n} e^{N\tilde{f}(u_i,x)} \prod_{1 \leq j < k \leq n} (u_j - u_k)^{4/\beta},$$

---

[2] Note however the presence of a misprint in [13] for the symplectic case.

Figure 3:   Asymptotic density (21) in the Hermite β-ensemble for $N = 8$ and $\beta = 2, 6, 10$.
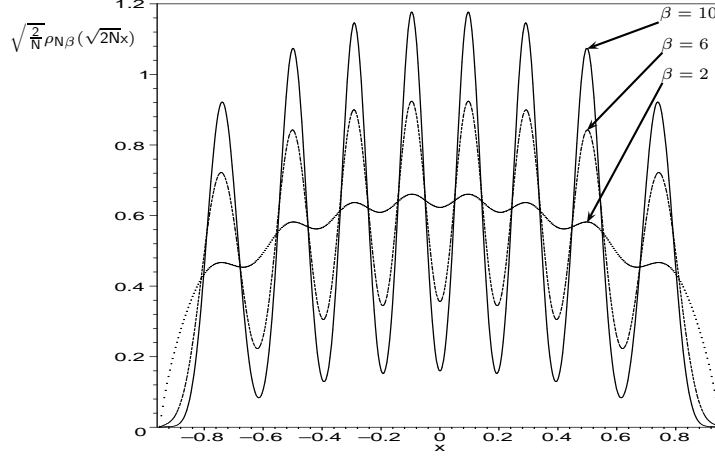
where $\tilde{f}(u, x) = f(u, x) - \dfrac{2(n-1)}{\beta N} \ln u$.

Proof. We first set $|u_j| = 1$, i.e., $u_j = e^{i\theta_j}$. $\mathcal{C}$ is such that θ_j goes from 0 to 2π. The integral is completely symmetric so that we have $n!$ possible arrangements of the type $0 \leq \theta_{i_1} \leq \ldots \leq \theta_{i_n} \leq 2\pi$. We choose $0 \leq \theta_n \leq \ldots \leq \theta_1 \leq 2\pi$. In that case,

$$\prod_{1 \leq j < k \leq n} |u_j - u_k|^{4/\beta} = \prod_{1 \leq j < k \leq n} \left(2 \sin \frac{\theta_i - \theta_j}{2} \right)^{4/\beta}.$$

The righthandside of the previous equation can be written as

$$\frac{1}{i^{2n(n-1)/\beta}} \prod_{i=1}^{n} u_i^{-2(n-1)/\beta} \prod_{1 \leq j < k \leq n} (u_i - u_j)^{4/\beta}.$$

We have proved that

$$\oint_{\mathcal{C}} du_1 \cdots \oint_{\mathcal{C}} du_n \prod_{i=1}^{n} e^{N f(u_i, x)} \prod_{1 \leq j < k \leq n} |u_j - u_k|^{4/\beta} =$$

$$\frac{n!}{(-1)^{n(n-1)/\beta}} \int_{\substack{|u_i|=1 \\ 0 \leq \arg(u_n) \leq \ldots \leq \arg(u_1) \leq 2\pi}} du_1 \cdots du_n \prod_{i=1}^{n} e^{N \tilde{f}(u_i, x)} \prod_{1 \leq j < k \leq n} (u_j - u_k)^{4/\beta}$$

The integrand is analytic everywhere but possibly on the segment $[0, 1]$. Therefore, we can apply Cauchy's theorem and deform the paths on the unit circle into any counterclockwise contours $\mathcal{C}_i$ around zero and starting at $u_i = 1$ as long as the ordering $0 \leq \arg(u_n) \leq \ldots \leq \arg(u_1) \leq 2\pi$ is satisfied. □

Proposition 5. *Let $\tilde{f}(u, x) = f(u, x) - (2 - 2/\beta) N^{-1} \ln u$, where $f(u, x)$ is the function appearing in the definition of $R_{N,\beta}(x)$, given in Eq. (7), and satisfying Eqs (13) and (14). Let also $\tilde{f}_\pm =$*

$\tilde{f}(u_\pm, x)$. *Suppose moreover that the saddle points are such that* $0 \leq \arg(u_+) < \arg(u_-) \leq 2\pi$. *Then,*

$$R_{N,\beta}(x) = -\binom{\beta}{\beta/2}(\Gamma_{\beta/2,\beta})^2(u_+ - u_-)^\beta \left(\frac{2}{NR}\right)^{\beta-1}$$

$$\times e^{\beta N(\tilde{f}_+ + \tilde{f}_-)/2} e^{\mathrm{i}(\beta-1)(\theta_+ + \theta_-)}\left[r_{N,\beta}(x) + \mathrm{O}(1/N)\right],$$

where

$$r_{N,\beta}(x) = 1 + 2\sum_{k=1}^{\lfloor\sqrt{\beta/2}\rfloor}\left[\left(\prod_{j=1}^{k}\frac{\Gamma(1+2j/\beta)}{\Gamma(1+2(j-k)/\beta)}\right)\right.$$

$$\left.\times\frac{e^{\mathrm{i}2k^2(\theta_+ + \theta_-)/\beta}}{(u_+ - u_-)^{4k^2/\beta}(NR)^{2k^2/\beta}}\cos\left(-\mathrm{i}kN(\tilde{f}_+ - \tilde{f}_-) + k(\theta_+ - \theta_-)(3 - 2/\beta)\right)\right].$$

Proof. We first use Lemma 4. The remaining steps are similar to those of Proposititon 2. $\square$

In reference [7], the density of the Laguerre β-ensemble has been written in terms of generalized hypergeometric functions [14]:

$$\rho_{N,\beta}(x) = N\frac{W_{a+2,\beta,N-1}}{W_{a,\beta,N}}x^{a\beta/2}e^{-\beta x/2}\ {}_1F_1{}^{(\beta/2)}(-N+1; a+2; t_1,\ldots,t_\beta)\Big|_{t_1=\ldots=t_\beta=x}.$$

There exist integral representations of the generalized hypergeometric functions [22]. For our purpose, the appropriate integral formula can be found in Chapter 11 of [9]; one easily shows that

$${}_1F_1{}^{(\alpha)}(-B; A+1+(n-1)/\alpha; t_1,\ldots,t_n)\Big|_{t_1=\ldots=t_n=x} =$$

$$\frac{\mathrm{i}^{2nB}}{M_n(A,B,1/\alpha)}\frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}}du_1\cdots\frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}}du_n\prod_{j=1}^{n}e^{xu_j}u_j^{-B-1}(1-u_j)^{A+B}\prod_{1\leq k<l\leq n}|u_k - u_l|^{2/\alpha},$$

where $\mathcal{C}$ is as previously described and where

$$M_n(A,B,C) = \prod_{j=1}^{n}\frac{\Gamma(1+A+B-C+jC)\Gamma(1+jC)}{\Gamma(1+A-C+jC)\Gamma(1+B-C+jC)\Gamma(1+C)}.$$

This implies that the Laguerre density in the bulk can be recast in an integral of the form (7):

$$\rho_{N,\beta}(4Nx) = \frac{N}{(2\pi\mathrm{i})^\beta}\frac{W_{a+2,\beta,N-1}}{W_{a,\beta,N}}\frac{(4Nx)^{a\beta/2}e^{-2\beta Nx}}{M_\beta(a+2/\beta-1, N-1, 2/\beta)}R_{N,\beta}(x) \tag{22}$$

provided that, in Eq. (7), $\mathcal{C}$ is a counterclockwise closed path around the origin and starting at $u = 1$, and

$$f(u,x) = 4xu - \ln u + \ln(1-u) + \frac{1}{N}\left(a - 2 + \frac{2}{\beta}\right)\ln(1-u). \tag{23}$$

Neglecting factors of order $1/N$, we see that this function has two simple saddle points,

$$u_\pm = \frac{1}{2}\left(1 \pm \mathrm{i}\sqrt{\frac{1}{x} - 1}\right),$$

which satisfy $\arg(u_+) < \arg(u_-)$ only if $0 < x < 1$. This implies

$$\frac{\partial^2}{\partial u^2}f(u,x)\Big|_{u_\pm} = Re^{\mathrm{i}\phi_\pm}, \qquad R = 16x^2\sqrt{\frac{1}{x} - 1}, \qquad \phi_\pm = \mp\frac{\pi}{2};$$

hence, the directions of steepest descent are $\theta_+ = 3\pi/4$ and $\theta_- = \pi/4$. We also have

$$\tilde{f}_\pm = 2x - \frac{a + 4/\beta - 4}{N}\ln 2\sqrt{x} \pm 2\mathrm{i}\left(x\sqrt{\frac{1}{x} - 1} - \arccos\sqrt{x}\right) \mp \mathrm{i}\frac{a}{N}\arccos\sqrt{x}.$$

The Stirling approximation readily gives

$$\frac{W_{a+2,\beta,N-1}}{W_{a,\beta,N}} = \left(\frac{\beta}{2}\right)^{1+a\beta}N^{a\beta/2}\frac{\Gamma(1 + \beta/2)}{\Gamma(1 + a\beta/2)\Gamma(1 + (a+1)\beta/2)}\left[1 + \mathrm{O}\left(\frac{1}{N}\right)\right]$$

Moreover, by using the Gauss multiplication formula, which reads

$$\prod_{j=0}^{n-1}\Gamma(z + j/n) = (2\pi)^{(n-1)/2}n^{1/2 - nz}\Gamma(nz),$$

one can show that

$$\binom{\beta}{\beta/2}\frac{(\Gamma_{\beta/2,\beta})^2}{M_\beta(a + \beta/2 - 1, N - 1, 2/\beta)}$$

$$= \pi^{\beta-1}\left(\frac{\beta}{2}\right)^{-1-a\beta}\frac{\Gamma(1 + a\beta/2)\Gamma(1 + (1+a)\beta/2)}{\Gamma(1 + \beta/2)N^{a\beta+2-\beta}}\left[1 + \mathrm{O}\left(\frac{1}{N}\right)\right]. \quad (24)$$

By substituting the above equations in Proposition 5, we obtain the following result.

Corollary 6. *Let $0 < x < 1$ and let $P_{\mathrm{MP}}(x)$ denote the (cumulative) probability distribution associated to the density ρ_{MP} defined in Eq. (6); i.e.,*

$$P_{\mathrm{MP}}(x) = \int_0^x \rho_{\mathrm{MP}}(t)dt = 1 + x\rho_{\mathrm{MP}}(x) - \frac{2}{\pi}\arccos x\,. \tag{25}$$
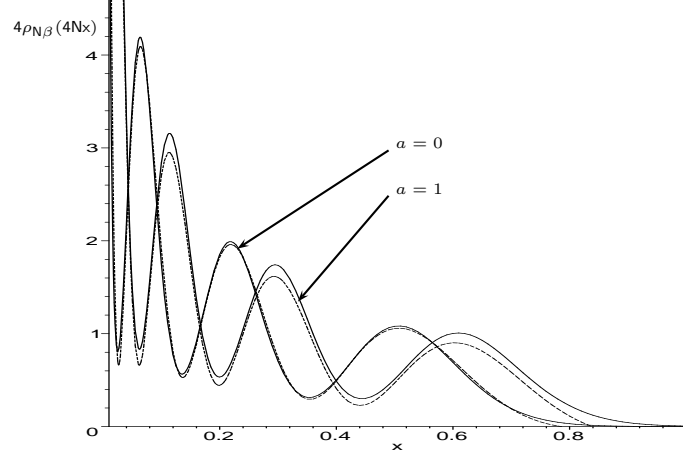
Then

$$4\rho_{N,\beta}(4Nx) = \rho_{\mathrm{MP}}(x)r_{N,\beta}(x) + \mathrm{O}\left(\frac{1}{N}\right),$$

where

$$r_{N,\beta}(x) = 1 + 2\sum_{k=1}^{\lfloor\sqrt{\beta/2}\rfloor}\frac{(-1)^k}{(2\pi^3x^2\rho_{\mathrm{MP}}(x)^3N)^{2k^2/\beta}}$$

$$\times\left(\prod_{j=1}^{k}\frac{\Gamma(1 + 2j/\beta)}{\Gamma(1 + 2(j-k)/\beta)}\right)\cos\left(2\pi kNP_{\mathrm{MP}}(x) + k\varphi(x,\beta)\right),$$

Figure 4: Comparison of the exact density (12), shown as a solid line, and the asymptotic density (27), shown as a dashed line, in the Hermite β-ensemble for $N = 4$, $\beta = 6$, and $a = 0, 1$.

$$\varphi(x, \beta) = \left(1 - \frac{2}{\beta}\right)\frac{\pi}{2} - 2a \arccos \sqrt{x}. \tag{26}$$

Again, we look at the first correction to the global density:

$$4\rho_{N,\beta}(4Nx) = \rho_{\mathrm{MP}}(x) + \mathrm{O}\left(\frac{1}{N}\right) + \mathrm{O}\left(\frac{1}{N^{8/\beta}}\right)$$
$$- \frac{2\Gamma(1 + 2/\beta)}{(2\pi^3 x^2 N)^{2/\beta}\rho_{\mathrm{MP}}(x)^{6/\beta - 1}} \cos\left(2\pi N P_{\mathrm{MP}}(x) + \varphi(x, \beta)\right). \tag{27}$$

Hence, neglecting the possible factor $\mathrm{O}(1/N)$, the first oscillatory corrections to the global density in the complex and quaternionionic Wishart ensembles are

$$4\rho_{N,\beta}(4Nx) - \rho_{\mathrm{MP}}(x) = \begin{cases} \dfrac{-1}{\pi^3 x^2 \rho_{\mathrm{MP}}(x)^2 N} \cos\left(2\pi N P_{\mathrm{MP}}(x) - 2a \arccos\sqrt{x}\right), & \beta = 2, \\[2ex] \dfrac{-1}{2^{1/2}\pi x \rho_{\mathrm{MP}}(x)^{1/2} N^{1/2}} \cos\left(2\pi N P_{\mathrm{MP}}(x) - 2a \arccos\sqrt{x} + \pi/4\right), & \beta = 4. \end{cases}$$

Contrary to the Gaussian unitary ensemble, the first correction in the complex Wishart ensemble has a non-null (when $a \neq 0$) correction of order $1/N$ which is not oscillatory [11]. Nevertheless, our oscillatory term is the same as the one given the latter reference. The $\beta = 4$ case has been recently studied in [10]; the dominant correction is purely oscillatory and is equal to the term given above.

Fig. 4 provides a numerical comparison between the asymptotic and the exact expressions of the density. Clearly, the asymptotic approximation is better for $a = 0$. Note that the oscillations shift to the right when a increases. In Fig. 5, we illustrate the effect of the variation of β on $4\rho_{N,\beta}(4Nx)$ for fixed N and a.

Figure 5: Asymptotic density (27) in the Laguerre β-ensemble for $N = 5$, $a = 0$, and $\beta = 4, 6, 8$.
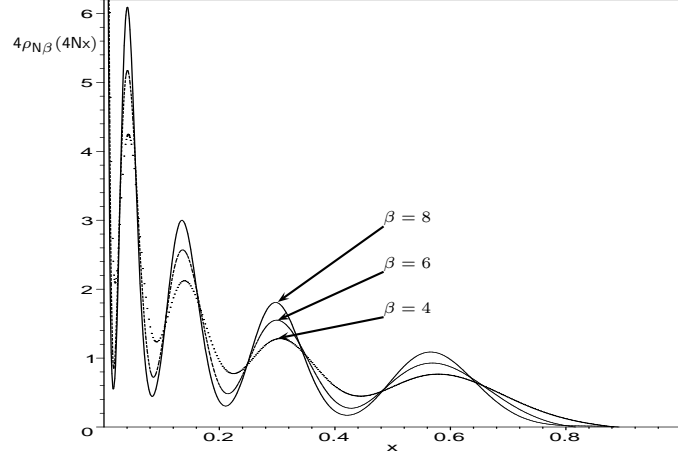
4   Density at the soft-edge

We have seen that the steepest descent method can be applied to the scaled densities only if $-1 < x < 1$ (Hermite case) or $0 < x < 1$ (Laguerre case). Indeed, when the spectral parameter x is outside these intervals, the contours of integration cannot be deformed into the steepest decent ones without transgressing the appropriate ordering of the variables of integration. A change of scaling is mandatory. The appropriate changes of variable at the soft edges have been obtained in [6]: the scaled densities $\rho_{N,\beta}(\sqrt{2N}x)$ (Hermite) and $\rho_{N,\beta}(4Nx)$ (Laguerre) should be replaced by $\rho_{N,\beta}(\sqrt{2N} + x/\sqrt{2}N^{1/3})$ (Hermite) and $\rho_{N,\beta}(4N + 2(2N)^{1/3}x)$ (Laguerre).

Technically, these new scalings make the two simple saddle points coalesce and become a double saddle point (or saddle point of order two). Then, the multiple Gaussian integrals are replaced by multiple Airy integrals, or integrals of the Kontsevich type [15]:

$$K_{n,\beta}(x) := -\frac{1}{(2\pi\mathrm{i})^n} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} dv_1 \cdots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} dv_n \prod_{j=1}^{n} e^{v_j^3/3 - xv_j} \prod_{1 \le k < l \le n} |v_k - v_l|^{4/\beta}. \tag{28}$$

We recall that the Airy function of a real variable x can be defined as follows:

$$\mathrm{Ai}(x) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} e^{v^3/3 - xv} dv, \tag{29}$$

and as a consequence

$$\mathrm{Ai}''(x) = x\mathrm{Ai}(x).$$

One readily verifies that

$$K_{1,\beta}(x) = -\mathrm{Ai}(x), \qquad K_{2,2}(x) = 2\big(\mathrm{Ai}'(x)^2 - x\mathrm{Ai}(x)^2\big).$$

It is worth mentioning that the function $K_{n,2}(x)$ has previously been studied in the context of the Gaussian unitary ensemble [8]. In particular, it has been shown that [3]

$$K_{n,2}(x) = -n! \det \left[\frac{d^{i+j-2}}{dx^{i+j-2}} \mathrm{Ai}(x) \right]_{i,j=1,\ldots,n}.$$
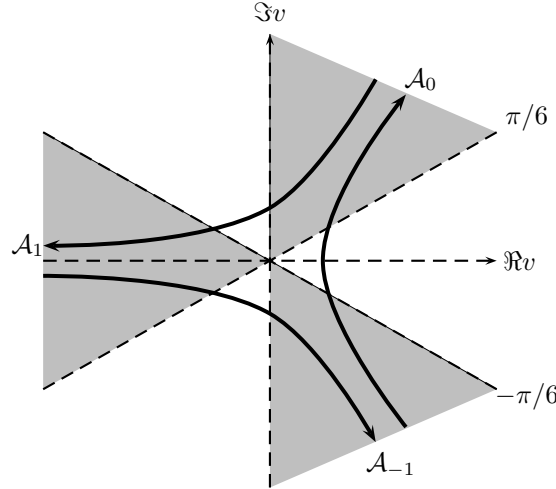
4.1 Hermite case

The next lemma generalizes a basic fact of the Airy function of a complex variable z; that is, the contour in Eq. (29) can be deformed so that

$$\mathrm{Ai}(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{A}_0} e^{v^3/3 - zv} dv, \tag{30}$$

where $\mathcal{A}_0$ is a simple path going from $\infty e^{\mathrm{i}\theta_a}$ to $\infty e^{\mathrm{i}\theta_b}$, where $-\pi/2 < \theta_a < -\pi/6$ and $\pi/6 < \theta_b < \pi/2$ (see in Fig. 6).

Figure 6: Possible contours in the integral representation of the Airy function.

Lemma 7. *Let $\mathcal{A}_0$ be the contour described above. Let also $\{\mathcal{V}_j\}$ denote a set of non-intersecting paths such that $\mathcal{V}_1 = \mathcal{A}_0$ and such that, for all $j \in \{2,\ldots,n\}$, $\mathcal{V}_j$ follows $\mathcal{A}_0$ but stop at v_{j-1}. Then*

$$K_{n,\beta}(x) := -\frac{n!}{(2\pi \mathrm{i})^n} \int_{\mathcal{V}_1} dv_1 \cdots \int_{\mathcal{V}_n} dv_n \prod_{j=1}^{n} e^{v_j^3/3 - xv_j} \prod_{1 \le k < l \le n} (v_k - v_l)^{4/\beta}, \tag{31}$$

where $-\pi < \arg v_j \le \pi$ and where $\arg (v_i - v_j)^{4/\beta} = 0$ when both $\Im v_i = 0 = \Im v_j$ and $\Re v_i > \Re v_j$.

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[3]Note however that the term $n!$ is missing in [8].

Proof. We essentially proceed as in Lemma 1. Firstly, we write Eq. (28) as ordered integrals along the imaginary axis and remove the absolute values. Secondly, the analyticity of the integrand and the property

$$\lim_{R \to \infty} \left[e^{v^3/3 - xv} \right]_{v = Re^{i\theta}} = 0 \quad \text{if} \quad -\frac{\pi}{2} < \theta < -\frac{\pi}{6}, \quad \frac{\pi}{6} < \theta < \frac{\pi}{2}, \quad \text{or} \quad \frac{5\pi}{6} < \theta < \frac{7\pi}{6} \quad (32)$$

are used to deform the contour of v_1 into $\mathcal{A}_0$. We finally complete the proof by exploiting the ordering of the variables, the choice of principal branch for $(v_i - v_j)^{4/\beta}$, and the Cauchy Theorem. $\square$

Proposition 8. *The integral $R_{N,\beta}$ defined by Eqs (7) and (18) satisfies*

$$R_{N,\beta} \left(1 + \frac{x}{2N^{2/3}} \right) = 4 \left(\frac{\pi}{2} \right)^{\beta} \frac{e^{\beta N/2 + \beta N^{1/3} x}}{2^{\beta N} N^{\beta - 2/3}} K_{\beta,\beta}(x) + O \left(\frac{1}{N^{\beta - 1/3}} \right).$$

Proof. According to Lemma 1, we have that

$$R_{N,\beta} \left(1 + \frac{x}{2N^{2/3}} \right) = \beta! \int_{\mathcal{C}_1} du_1 \cdots \int_{\mathcal{C}_\beta} du_\beta \prod_{i=1}^{\beta} e^{N f(u_i, 1 + x/2N^{2/3})} \prod_{1 \le j < k \le \beta} (u_j - u_k)^{4/\beta}.$$

Considering $x/2N^{2/3} \ll 1$,

$$f\left(u, 1 + \frac{x}{2N^{2/3}} \right) = g(u, x) + \frac{x}{2(iu + x)N^{2/3}} + O \left(\frac{1}{N^{4/3}} \right),$$

where

$$g(u, x) = -2u^2 + \ln(iu + x).$$

This functions has a double saddle point $u_0 = i/2$:

$$\left. \frac{\partial}{\partial u} g(u, x) \right|_{u = u_0} = 0 = \left. \frac{\partial^2}{\partial u^2} g(u, x) \right|_{u = u_0}, \qquad \left. \frac{\partial^3}{\partial u^3} g(u, x) \right|_{u = u_0} = Re^{i\phi_0} = 16 e^{-i\pi/2}.$$

The directions for which the decrease of f is maximum are determined by both conditions $\cos(3\theta_0 + \phi_0) = -1$ and $\sin(3\theta_0 + \phi_0) = 0$. Thus, the angles of steepest descent are $\theta_0 = -5\pi/6, -\pi/6, \pi/2$. We choose the two former and make the following change of variables:

$$v_j = 2i N^{1/3} (u_j - u_0).$$

This implies that

$$N f\left(u_j, 1 + \frac{x}{2N^{2/3}} \right) = \frac{N}{2} - N \ln 2 + N^{1/3} x + \ln 2 + \frac{1}{3} v_j^3 - x v_j + O \left(\frac{1}{N^{1/3}} \right).$$

Let $\{\mathcal{V}_j\}$ denote the set of ordered and non-intersecting contours of steepest descent: $\mathcal{V}_1$ starts at $\infty e^{-i\pi/3}$, passes through the origin and stops at $\infty e^{i\pi/3}$; $\mathcal{V}_j$ follows $\mathcal{V}_1$ but stops at v_{j-1}, where $j = 2, \ldots, \beta$. We have proved that

$$R_{N,\beta} \left(1 + \frac{x}{2N^{2/3}} \right) = -4\beta! \frac{e^{\beta N/2 + \beta N^{1/3} x}}{2^{\beta(N+2)} N^{\beta - 2/3} i^{3\beta}}$$

$$\times \int_{\mathcal{V}_1} dv_1 \cdots \int_{\mathcal{V}_\beta} dv_\beta \prod_{i=1}^{\beta} e^{v_i^3/3 - x v_i} \prod_{1 \le j < k \le \beta} (v_j - v_k)^{4/\beta} + O \left(\frac{1}{N^{1/3}} \right).$$

Consequently, we can apply Lemma 7 and the proposition follows. $\square$

Let us go back to Eq. (17) and make the change of scaling $x \mapsto 1 + x/(2N^{2/3})$:

$$\rho_{N,\beta}\left(\sqrt{2N} + \frac{x}{\sqrt{2N^{1/3}}}\right)$$
$$= \frac{1}{2}\frac{G_{\beta,N-1}}{G_{\beta,N}\Gamma_{\beta,\beta}}(2N)^{\beta N/2+\beta}e^{-\beta N}e^{-\beta N^{1/3}x}e^{-\beta x^2/(4N^{1/3})}R_{N,\beta}\left(1 + \frac{x}{2N^{2/3}}\right).$$

Proposition 8 and some manipulations directly imply the following.

Corollary 9. *The density in the Hermite β-ensemble evaluated at soft edge is proportional to the an integral of the Kontsevich type:*

$$\frac{1}{\sqrt{2N^{1/3}}}\rho_{N,\beta}\left(\sqrt{2N} + \frac{x}{\sqrt{2N^{1/3}}}\right)$$
$$= \frac{1}{2\pi}\left(\frac{4\pi}{\beta}\right)^{\beta/2}\frac{\Gamma(1+\beta/2)}{\prod_{j=2}^{\beta}\Gamma(1+2/\beta)^{-1}\Gamma(1+2j/\beta)}K_{\beta,\beta}(x) + O\left(\frac{1}{N^{1/3}}\right).$$

4.2 Laguerre case

It is obvious from Eq. (30) and Fig. 6 that

$$\frac{1}{2\pi i}\int_{\mathcal{A}_{-1}\cup\mathcal{A}_0\cup\mathcal{A}_1}e^{v^3/3-zv}dv = 0 \quad\Longleftrightarrow\quad e^{-i2\pi/3}\mathrm{Ai}(e^{-i2\pi/3}z)+\mathrm{Ai}(z)+e^{i2\pi/3}\mathrm{Ai}(e^{i2\pi/3}z) = 0.$$

This result can be extended to the Kontsevich type integral as follows.

Lemma 10. *Let $\mathcal{A}_{-1}, \mathcal{A}_1$ be the contours depicted in Fig. 6. Let also $\{\widetilde{\mathcal{V}}_j\}$ denote a set of non-intersecting paths such that $\widetilde{\mathcal{V}}_1 = \mathcal{A}_{-1}\cup\mathcal{A}_1$ and such that, for all $j \in \{2,\ldots,n\}$, $\widetilde{\mathcal{V}}_j$ follows $\widetilde{\mathcal{V}}_1$ but stops at v_{j-1}. Then*

$$K_{n,\beta}(x) = (-1)^{n+1}\frac{n!}{(2\pi i)^n}\int_{\widetilde{\mathcal{V}}_1}dv_1\cdots\int_{\widetilde{\mathcal{V}}_n}dv_n\prod_{j=1}^{n}e^{v_j^3/3-xv_j}\prod_{1\leq k<l\leq n}(v_k - v_l)^{4/\beta}, \qquad (33)$$

where $-\pi < \arg v_j \leq \pi$ and where $\arg(v_i - v_j)^{4/\beta} = 0$ when both $\Im v_i = 0 = \Im v_j$ and $\Re v_i > \Re v_j$.

Proof. By virtue of Eq. (32), it is possible to join $\mathcal{A}_{-1}$ and $\mathcal{A}_1$. Thus, Cauchy's Theorem implies that $\widetilde{\mathcal{V}}_1$ can be replaced $-\mathcal{A}_0$. The constraint on the ordering of the variables $\{v_2,\ldots,v_n\}$ is then considered and equivalence between Eq. (31) and Eq. (33) follows. $\square$

Proposition 11. *The integral $R_{N,\beta}$ defined by Eqs (7) and (23) satisfies*

$$R_{N,\beta}\left(1 + \frac{x}{(2N)^{2/3}}\right) = (2\pi i)^{\beta}\frac{e^{2\beta N}e^{\beta(2N)^{1/3}x}}{2^{a\beta+4/3}N^{\beta-2/3}}K_{\beta,\beta}(x) + O\left(\frac{1}{N^{\beta-1/3}}\right).$$

Proof. We essentially follow the proof of Proposition 8. By virtue of Lemma 4, we have that

$$R_{N,\beta}\left(1 + \frac{x}{(2N)^{2/3}}\right) = \beta!\oint_{\mathcal{C}_1}du_1\cdots\oint_{\mathcal{C}_\beta}du_\beta\prod_{i=1}^{\beta}e^{N\tilde{f}(u_i,1+x/(2N)^{2/3})}\prod_{1\leq j<k\leq\beta}(u_j - u_k)^{4/\beta},$$

where

$$\tilde{f}\left(u, 1 + \frac{x}{(2N)^{2/3}}\right) = g(u) + \frac{1}{(2N)^{2/3}}xu + \frac{1}{N}\left(a - 2 + \frac{2}{\beta}\right)\ln(1-u) + \frac{1}{N}\left(\frac{2}{\beta} - 2\right)\ln(u),$$

for

$$g(u) = 4u + \ln(1-u) - \ln(u).$$

The latter function possesses a double saddle point $u_0 = 1/2$. One can check that

$$d^3 g(u)/du^3|_{u=u_0} = 32e^{i\pi},$$

so the steepest descent angles are $\theta_0 = 0, 2\pi/3, 4\pi/3$. The contour of u_1 is chosen such that: (1) it approaches u_0 by following the real axis in the negative direction; (2) it leaves the saddle point with an angle $2\pi/3$; (3) it turns around the origin in the positive direction; (4) it comes back to the u_0 with an argument of $4\pi/3$; (5) it finally leaves this point and reaches the point $u = 1$ by following the real axis. The ordering of the variables around the origin implies moreover that u_i follows u_1 but stops at u_{i-1}. When N is large, step (3) is irrelevant and the steepest descent contour brakes into two disjoints paths, namely (1)-(2) and (4)-(5). Now we set

$$\tilde{v}_j = 2(2N)^{1/3}(u_0 - u_j),$$

which means that

$$N\tilde{f}\left(u_j, 1 + \frac{x}{(2N)^{2/3}}\right) = 2N + (2N)^{1/3}x + \left(4 - \frac{4}{\beta} - a\right)\ln 2 + \frac{1}{3}\tilde{v}_j^3 - x\tilde{v}_j + \mathrm{O}\left(\frac{1}{N^{1/3}}\right).$$

When $N \to \infty$, the contours of the variables $\{\tilde{v}_j\}$ that give the major contribution to the integral, denoted by $\{\tilde{\mathcal{V}}_j\}$, behave as follows: $\tilde{\mathcal{V}}_1$ is the union of the path that begins at $-\infty$, passes close to the origin and ends at $\infty e^{-i\pi/3}$ together with the path that starts at $\infty e^{i\pi/3}$, goes near the origin and stops at $-\infty e^{-i\pi/3}$; $\tilde{\mathcal{V}}_j$ follows $\tilde{\mathcal{V}}_1$ but stops at v_{j-1}, where $j = 2, \ldots, \beta$. Therefore,

$$R_{N,\beta}\left(1 + \frac{x}{(2N)^{2/3}}\right) = \beta! \frac{e^{2\beta N}e^{\beta(2N)^{1/3}x}}{2^{a\beta+4/3}N^{\beta-2/3}}$$

$$\times \int_{\tilde{\mathcal{V}}_1} d\tilde{v}_1 \cdots \int_{\tilde{\mathcal{V}}_\beta} d\tilde{v}_\beta \prod_{i=1}^{\beta} e^{\tilde{v}_i^3/3 - x\tilde{v}_i} \prod_{1 \le j < k \le \beta} (\tilde{v}_j - \tilde{v}_k)^{4/\beta} + \mathrm{O}\left(\frac{1}{N^{1/3}}\right).$$

Lemma 10 finally provides the sought for result. □

We apply the last proposition to the scaled expression of the Laguerre density given in (22):

$$\rho_{N,\beta}\left(4N + 2(2N)^{1/3}x\right)$$

$$= \frac{N}{(2\pi i)^\beta} \frac{W_{a+2,\beta,N-1}}{W_{a,\beta,N}} \frac{(4N)^{a\beta/2}e^{-2\beta N}e^{-\beta(2N)^{1/3}}}{M_\beta(a + 2/\beta - 1, N - 1, 2/\beta)} R_{N,\beta}\left(1 + \frac{x}{(2N)^{2/3}}\right).$$

Minor manipulations and use of Stirling's approximation give the sought limiting soft edge density, which is identical to that obtained in Corollary 9 for the Hermite β-ensemble.

Corollary 12. *The density in the Laguerre β-ensemble evaluated at the soft edge is proportional to the an integral of the Kontsevich type:*

$$2(2N)^{1/3}\rho_{N,\beta}\left(4N + 2(2N)^{1/3}x\right)$$
$$= \frac{1}{2\pi}\left(\frac{4\pi}{\beta}\right)^{\beta/2}\frac{\Gamma(1+\beta/2)}{\prod_{j=2}^{\beta}\Gamma(1+2/\beta)^{-1}\Gamma(1+2j/\beta)}K_{\beta,\beta}(x) + \mathrm{O}\left(\frac{1}{N^{1/3}}\right).$$

5 Asymptotics of the Kontsevich type integral

Here we consider the leading order of $K_{\beta,\beta}(x)$ when $x \to \pm\infty$. This allows to match the soft edge density with the bulk density expanded about the edge.

Proposition 13. *When x is large and positive*

$$K_{\beta,\beta}(x) = \frac{\Gamma_{\beta,\beta}}{(2\pi)^{\beta}}\frac{e^{-\frac{2\beta}{3}x^{3/2}}}{x^{3\beta/4-1/2}} + \mathrm{O}\left(\frac{1}{x^{3\beta/4+1}}\right).$$

Proof. Following the discussion in the proof of Proposition 8, we first change the contours in the Kontsevich like integral:

$$K_{\beta,\beta}(x) = -\frac{\beta!}{(2\pi\mathrm{i})^{\beta}}\int_{\mathcal{V}_1}dv_1\cdots\int_{\mathcal{V}_\beta}dv_\beta\prod_{i=1}^{\beta}e^{v_j^3/3-xv_j}\prod_{1\leq j<k\leq\beta}(v_j-v_k)^{4/\beta},$$

where $\{\mathcal{V}_j\}$ is such that $\mathcal{V}_1$ goes from $\infty e^{-\mathrm{i}\theta}$ to $\infty e^{\mathrm{i}\theta}$, passing through the point $\sqrt{x}$, and such that $\mathcal{V}_j$ goes from $\infty e^{-\mathrm{i}\theta}$ to v_{j-1} for all $j = 2,\ldots,\beta$ and $\pi/6 < \theta < \pi/2$. We now set

$$w_j = x^{1/4}(v_j - x^{1/2})e^{-\mathrm{i}\pi/2}.$$

Thus,

$$K_{\beta,\beta}(x) = \frac{\beta!}{(2\pi)^{\beta}}\frac{e^{-2\beta x^{3/2}/3}}{x^{3\beta/4-1/2}}\int_{\mathcal{W}_1}dw_1\cdots\int_{\mathcal{W}_\beta}dw_\beta\prod_{i=1}^{\beta}e^{-w_j^2-\mathrm{i}w_j^3/3x^{3/4}}\prod_{1\leq j<k\leq\beta}(w_j-w_k)^{4/\beta},$$

where $\mathcal{W}_j = e^{-\mathrm{i}\pi/2}\mathcal{V}_j$. By virtue of Lemma 1, we can write

$$K_{\beta,\beta}(z) = \frac{1}{(2\pi)^{\beta}}\frac{e^{-2\beta x^{3/2}/3}}{x^{3\beta/4-1/2}}\int_{-\infty}^{\infty}dw_1\cdots\int_{-\infty}^{\infty}dw_\beta\prod_{i=1}^{\beta}e^{-w_j^2+\mathrm{O}(x^{-3/4})}\prod_{1\leq j<k\leq\beta}|w_j-w_k|^{4/\beta}.$$

Note that the term $\mathrm{O}(x^{-3/4})$ is odd in w_j. As explained in the proof of Proposition 17, this implies that the actual correction to the integral is of order $x^{-3/2}$. We finally obtain the desired expression by comparing the last equation with Eq. (15). $\square$

Applying Proposition 13, the next result gives the behavior of the density when the spectral parameter leaves the bulk.

Corollary 14. *Let $\sigma(x)$ denote the density evaluated at the soft edge:*

$$\sigma(x) = \begin{cases} \lim\limits_{N\to\infty} \dfrac{1}{\sqrt{2}N^{1/3}} \rho_{N,\beta}\left(\sqrt{2N} + \dfrac{x}{\sqrt{2}N^{1/3}}\right), & \text{(Hermite)} \\ \lim\limits_{N\to\infty} 2(2N)^{1/3} \rho_{N,\beta}\left(4N + 2(2N)^{1/3}x\right), & \text{(Laguerre).} \end{cases}$$

Then, as $x \to \infty$,

$$\sigma(x) = \frac{1}{2\pi} \frac{\Gamma(1+\beta/2)}{(4\beta)^{\beta/2}} \frac{e^{-\frac{2\beta}{3}x^{3/2}}}{x^{3\beta/4-1/2}} + \mathrm{O}\left(\frac{1}{x^{3\beta/4+1}}\right).$$

When the density is evaluated at points inside the bulk but close to the edge, we should observe both decrease and oscillation (see Fig. 2-5). This is confirmed in next paragraphs.

Proposition 15. *Let $x = -|x|$. When $|x|$ is large,*

$$K_{\beta,\beta}(x) = \frac{(\Gamma_{\beta/2,\beta})^2}{\pi^\beta}\binom{\beta}{\beta/2}\sqrt{|x|}\, k_{x,\beta} + \mathrm{O}\left(\frac{1}{x^{5/2}}\right),$$

where

$$k_{x,\beta} = 1+2\sum_{k=1}^{\lfloor\sqrt{\beta/2}\rfloor} \frac{(-1)^k}{2^{6k^2/\beta}|x|^{3k^2/\beta}}\left(\prod_{j=1}^k \frac{\Gamma(1+2j/\beta)}{\Gamma(1+2(j-k)/\beta)}\right)\cos\left(\frac{4k}{3}|x|^{3/2} - \frac{\pi}{2}k\left(1-\frac{2}{\beta}\right)\right).$$

Proof. By rescaling the variables in Eq. (28), we get

$$K_{\beta,\beta}(x) = -\frac{|x|^{3\beta/2-1}}{(2\pi\mathrm{i})^\beta} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} dv_1 \cdots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} dv_\beta \prod_{j=1}^\beta e^{|x|^{3/2}f(v_j)} \prod_{1\le k<l\le\beta} |v_k - v_l|^{4/\beta},$$

where

$$f(v) = \frac{1}{3}v^3 + v.$$

The function f possesses two simple saddle points, namely, $v_\pm = e^{\pm\mathrm{i}\pi/2}$. We have $f_\pm = f(v_\pm) = \pm 2\mathrm{i}/3$ and $f''(v_\pm) = 2e^{\pm\mathrm{i}\pi/2}$; whence the angle of steepest descent are $\theta_\pm = \pi/2 \mp \pi/4$. The remainder of the proof is a straightforward application of Proposition 2. $\square$

Corollary 16. *Let $\sigma(x)$ be the quantity defined in Lemma 14. When $x \to -\infty$, we have*

$$\pi\sigma(x) = \sqrt{|x|} - \frac{\Gamma(1+\beta/2)}{2^{6/\beta-1}|x|^{3/\beta-1/2}}\cos\left(\frac{4}{3}|x|^{3/2} - \frac{\pi}{2}\left(1-\frac{2}{\beta}\right)\right)$$

$$+ \mathrm{O}\left(\frac{1}{|x|^{5/2}}\right) + \mathrm{O}\left(\frac{1}{|x|^{6/\beta-1/2}}\right).$$

The previous result can be obtained directly from the asymptotic density in the bulk of the Hermite (or Laguerre) β-ensemble. Indeed, the change of variable $x \mapsto 1 - |x|/(2N^{2/3})$ in Eq. (20) and the development of this expression for $N^{-1/3}|x| \ll 1$ reclaims Corollary 16. However, it is impossible to derive Corollary 14 from Eq. (20) by such an expansion. Note finally that Corollaries 14-16 imply that the density at the soft edge of the Laguerre β-ensemble is independent of a when both N and $|x|$ are large.

6   Concluding remarks

The aim of the article was to determine the large-N asymptotic expansion of the density in the Hermite and Laguerre β-ensembles when $\beta \in 2\mathbb{N}$.

We have shown that the first correction to the global density is purely oscillatory when $\beta > 2$ and is of order $N^{2/\beta}$. In the Hermite ensemble of $N \times N$ random matrices, the density contains N peaks; the greater is β and the higher are the oscillations. The influence of the Dyson parameter on the oscillations is the same in the Laguerre ensemble. However, the density in the latter ensemble contains $N - 1$ summits and a (delta) divergence at the origin.

These results agree with the large-β asymptotic analysis realized recently in [4]. More precisely, it has been proved that for $\beta \to \infty$, the density in the bulk of the Hermite ensemble can be written as a sum of N Gaussian distributions centered at the zeros of an Hermite polynomial of degree N (and similarly for the Laguerre case). These conclusions are, of course, coherent with the log-gas analogy presented in Section 1. Note that no constraints on β are imposed in [4]. Consequently, we may surmise that our asymptotic formulas (21) and (27) are valid for any real β greater that 2, though the general method to prove this is still missing.

We have also shown that the density of the Hermite and Laguerre ensembles are both proportional to a Kontsevich like integral $K_{\beta,\beta}(x)$ when evaluated about the edges of the spectrum. Although the exact densities of the Hermite and Laguerre ensembles are quite different, the asymptotic analysis of $K_{\beta,\beta}(x)$ has revealed that they approach the same function in the soft edge scaling, thus verifying the expected universality. The Kontsevich like integral itself is a special function generalizing the Airy integral and, as such, is worthy for independent study.

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